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COMMON ACUTE POINTS AND CONVERGENCE THEOREMS FOR FAMILIES OF NONLINEAR MAPPINGS

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ABSTRACT. In this paper, we prove an attractive points theorem and strong convergence theorems of Halpern's type [21] for uniformly asymptotically regular λ-hybrid mappings in a star-shaped subset of a Hilbert space. Using these results, we obtain a fixed point theorem and some strong convergence theorems. Further, we prove convergence theorems by using the concept of acute points of nonlinear mappings.

1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and let C be a nonempty subset of H. For a mapping $T : C \to C$, we denote by $F(T)$ the set of *fixed points* of T and by $A(T)$ the set of *attractive points* [29] of T, i.e.,

(i) $F(T) = \{z \in C : Tz = z\}$;
(ii) $A(T) = \{z \in H : \|Tx - z\| \leq \|x - z\|,\ \forall x \in C\}$.

A mapping $T : C \to C$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

In 1975, Baillon [14] proved the following first nonlinear ergodic theorem in a Hilbert space: Let C be a nonempty bounded closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself. Then, for any $x \in C$, $S_n x = \frac{1}{n}\sum_{i=0}^{n-1} T^i x$ converges weakly to a fixed point of T (see also [27]).

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Kocourek, Takahashi and Yao [23] introduced a broad class of nonlinear mappings called *generalized hybrid* which containing nonexpansive mappings, nonspreading mappings, and hybrid mappings in a Hilbert space. They proved a mean convergence theorem for generalized hybrid mappings which generalizes Baillon's nonlinear ergodic theorem [14]. Aoyama, Iemoto, Kohsaka and Takahashi [4] introduced the class of λ-hybrid mappings in a Hilbert space. This class obtain the classes of nonexpansive mappings, nonspreading mappings, and hybrid mappings in a Hilbert space. They proved fixed point theorems and mean convergence theorems for such mappings. Motivated by Baillon [14], and Kocourek, Takahashi and Yao [23], Takahashi and Takeuchi [29] introduced the concept of attractive points of a nonlinear mapping in a Hilbert space and they proved a mean convergence theorem of Baillon's type without convexity for generalized hybrid mappings. In 1992, Wittmann [30] proved the following strong convergence theorems of Halpern's type [21] in a Hilbert space;

Theorem 1.1. *Let C be a nonempty closed convex subset of a Hilbert space H. Let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. For any $x_1 = x \in C$, define a sequence $\{x_n\}$ in C by*

$$x_{n+1} = \alpha_n x + (1-\alpha_n) T x_n, \ \forall n \geq 1$$

where $\{\alpha_n\} \subset [0,1]$ satisfies

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty.$$

Then, $\{x_n\}$ converges strongly to $P_{F(T)}x$, where $P_{F(T)}$ is the metric projection from H onto $F(T)$.

Motivated by Takahashi and Takeuchi [29], Akashi and Takahashi [2] proved a strong convergence theorem of Halpern's type [21] for nonexpansive mappings in a star-shaped subset of a Hilbert space. On the other hand, Domingues Benavides, Acedo and Xu [18] proved strong convergence theorems of Halpern's type [21] for uniformly asymptotically regular one-parameter nonexpansive semigroups. The author [8] studied Halpern's type iterations for nonexpansive semigroups and proved strong convergence theorems for uniformly asymptotically regular nonexpansive semigroups in Hilbert spaces (see also [1, 7, 9, 18, 26, 27]).

In this paper, we prove an attractive points theorem and strong convergence theorems of Halpern's type [21] for uniformly asymptotically regular λ-hybrid mappings in a star-shaped subset of a Hilbert space. Using these results, we obtain a fixed point theorem and some strong convergence theorems. Further, we prove convergence theorems by using the concept of acute points of nonlinear mappings.

2. Preliminaries and notations

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$ the set of all positive integers and the set of all real numbers, respectively. We also denote by $\mathbb{Z}^+$ and $\mathbb{R}^+$ the set of all nonnegative integers and the set of all nonnegative real numbers, respectively. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. We know the following basic equality from [27]. For $x, y \in H$ and $\lambda \in \mathbb{R}$, we have

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y \rangle \tag{2.1}$$

and

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2. \tag{2.2}$$

Furthermore, we obtain that for all $x, y, w \in H$,

$$\langle (x-y) + (x-w), y-w \rangle = \|x-w\|^2 - \|x-y\|^2. \tag{2.3}$$

In fact, we have that

$$\begin{aligned}
&\langle (x-y) + (x-w), y-w \rangle \\
&= \langle (x-y) + (x-w), (y-x) + (x-w) \rangle \\
&= \|x-w\|^2 - \|x-y\|^2 + \langle x-y, x-w \rangle + \langle x-w, y-x \rangle \\
&= \|x-w\|^2 - \|x-y\|^2.
\end{aligned}$$

Let C be a closed and convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|$$

for all $y \in C$. The mapping P_C is called the *metric projection* of H onto C. It is characterized by

$$\langle P_C x - y, x - P_C x \rangle \geq 0$$

for all $y \in C$. See [27] for more details. The following result is well-known (see [27]).

Lemma 2.1. *Let C be a nonempty, bounded, closed and convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself. Then, $F(T) \neq \emptyset$.*

We write $x_n \to x$ (or $\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in H converges strongly to x. We also write $x_n \rightharpoonup x$ (or w-$\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in H converges weakly to x. In a Hilbert space, it is well known that $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ imply $x_n \to x$.

A mapping $T : C \to C$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. Let $\lambda \in \mathbb{R}$ be given. Following [4], we say that a mapping $T : C \to C$ is λ-hybrid if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2(1 - \lambda)\langle x - Tx, y - Ty\rangle$$

for all $x, y \in C$. It is obvious that T is 1-hybrid if and only if T is nonexpansive; T is 0-hybrid if and only if T is nonspreading [24]; T is 1/2-hybrid if and only if T is hybrid [28]); If $\lambda > 1$, then T is λ-hybrid if and only if $T = I$. It is known [3, Proposition 2.2] that if $\lambda < 2$ and $\alpha = (1-\lambda)/(2-\lambda)$, then T is λ-hybrid if and only if it is α-nonexpansive [3], i.e.,

$$\|Tx - Ty\|^2 \leq \alpha(\|x - Ty\|^2 + \|Tx - y\|^2 + (1 - 2\alpha)\|x - y\|^2$$

for all $x, y \in C$. In general, nonspreading and hybrid mappings are not continuous mappings. A mapping $T : C \to C$ is called *quasi-nonexpansive* if $F(T)$ is nonempty and $\|w - Tx\| \leq \|w - y\|$ for all $w \in F(T)$ and $x \in C$. By Dotson [17, Theorem 1] and Ithoh and Takahashi [22, Corollary 1], we know that $F(T)$ is closed convex whenever T is quasi-nonexpansive. Every λ-hybrid with a fixed point is cleary quasi-nonexpansive. Thus, the set of fixed point of each λ-hybrid mapping is closed convex. The mapping T is said to be firmly nenexpansive if

$$\|Tx - Ty\|^2 + \|(I - T)x - (I - T)y\|^2 \leq \|x - y\|^2$$

for all $x, y \in C$(see [15, 16, 19, 20]. It is known [4, Lemma 3.1] that if T is firmly nenexpansive, then T is λ-hybrid for each $\lambda \in [0, 1]$.

3. Lemmas

In this section, we give some lemmas which are used in the proofs of our main theorems. We have basic properties of attractive points of nonlinear mappings in a Hilbert space (see [29]).

Lemma 3.1 ([29]). *Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H. Let T be a mappings of C into itself. If $A(T) \neq \emptyset$, then $F(T) \neq \emptyset$.*

Lemma 3.2 ([29]). *Let H be a Hilbert space, let C be a nonempty subset of H. Let T be a mappings of C into H. Then, $A(T)$ is a closed and convex subset of H.*

We also have the following lemma (see also [12, 29]).

Lemma 3.3 ([29]). *Let H be a Hilbert space, let C be a nonempty subset of H. Let T be a mappings of C into H. Let $\{u_n\}$ be a sequence in H such that*

$$\varlimsup_{n\to\infty} \langle (u_n - y) + (u_n - Ty), y - Ty \rangle \leq 0$$

for all $y \in C$. If a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ converges weakly to $u \in H$, then $u \in A(T)$.

To prove our main results, we need the following lemma (see [5]; see also [31]).

Lemma 3.4. *Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence of $[0,1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$. Let $\{\beta_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \beta_n < \infty$ and let $\{\gamma_n\}$ be a sequence of real numbers with $\varlimsup_{n\to\infty} \gamma_n \leq 0$. Suppose that*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n \gamma_n + \beta_n$$

for all $n \in \mathbb{N}$. Then, $\lim_{n\to\infty} s_n = 0$.

4. Acute points and convergence theorems

In this section, we prove convergence theorems by using the concept of k-acute points of a mapping for $k \in [0,1]$. Let C be a subset of a Hilbert space H and let T be a mapping of C into H. A mapping T is said to be L-Lipschitzian if $\|Tx - Ty\| \leq L\|x - y\|$ for any $x, y \in C$, where $L \in [0, \infty)$. Usually, T is said to be quasi-nonexpansive if

$$(1)\ F(T) \neq \emptyset, \quad (2) \quad \|Tx - v\| \leq \|x - v\| \quad \text{for} \quad x \in C, v \in F(T).$$

Let I be the identity mapping on C. Usually, T is said to be hemi–contractive if

$$(1) F(T) \neq \emptyset, \quad (2) \|Tx-v\|^2 \leq \|x-v\|^2+\|x-Tx\|^2 \quad \text{for } x \in C,\ v \in F(T).$$

These concepts depend on the condition $F(T) \neq \emptyset$. Usually, T is said to be k-demi-contractive if

$$(1) \quad F(T) \neq \emptyset, \quad (2) \quad \|Tx-v\|^2 \leq \|x-v\|^2+k\|x-Tx\|^2 \quad \text{for} \quad x \in C,\ v \in F(T).$$

We also call T a demi–contraction if T is a k-demi–contraction for some $k \in [0,1)$. Assume $F(T) \neq \emptyset$.

Let $k \in [0,1]$. We define the set of k–acute points $\mathcal{A}_k(T)$ of T by

$$\mathcal{A}_k(T) = \{\, v \in H : \ \|Tx-v\|^2 \leq \|x-v\|^2+k\|x-Tx\|^2 \quad \text{for all } x \in C \,\}.$$

We denote $\mathcal{A}_0(T)$ by $A(T)$ because $\mathcal{A}_0(T)$ and attractive points set of T are the same. We denote $\mathcal{A}_1(T)$ by $\mathcal{A}(T)$, that is,

$$\mathcal{A}(T) = \{\, v \in H : \ \|Tx-v\|^2 \leq \|x-v\|^2+\|x-Tx\|^2 \quad \text{for all } x \in C \,\}.$$

Now, we get the following convergence theorems [13]. We consider weak convergence theorems in the case $A(S) \neq \emptyset$ and $F(S) \subset \mathcal{A}(S)$. To have the following results, we have to assume demicloseness at 0 of $I - S$.

Theorem 4.1 ([13]). *Let $a, b \in (0,1)$ with $a \leq b$ and $\{a_n\}$ be a sequence in $[a,b]$. Let C be a weakly closed subset of a Hilbert space H. Let S be a self–mapping on C such that $F(S) \subset \mathcal{A}(S)$, $A(S) \neq \emptyset$, and $I - S$ is demiclosed at 0. Suppose there is a sequence $\{u_n\}$ in C such that*

$$u_{n+1} = a_n u_n + (1-a_n) S u_n \qquad \textit{for } n \in \mathbb{N}.$$

Then, $\{u_n\}$ converges weakly to some $u \in F(S)$.

Theorem 4.2 ([13]). *Let $a, b \in (0,1)$ with $a \leq b$ and $\{a_n\}$ be a sequence in $[a,b]$. Let C be a weakly closed subset of a Hilbert space H and T be a self-mapping on C such that $I - T$ is demiclosed at 0. Assume that one of the followings hold.*

(1) T is hemi-contractive with $A(T) \neq \emptyset$. S is the mapping defined by $S = T$.

(2) T is k-demi–contractive. S is the mapping defined by $S = kI + (1-k)T$.

(3) T is quasi–nonexpansive. S is the mapping defined by $S = T$.

Suppose S is a self-mapping on C and there is a sequence $\{u_n\}$ in C such that

$$u_{n+1} = a_n u_n + (1 - a_n) S u_n \qquad \text{for } n \in \mathbb{N}.$$

Then, $\{u_n\}$ converges weakly to some $u \in F(T)$.

Now, we get a nonlinear mean ergodic theorem (see also [14]).

Theorem 4.3 ([13]). *Let $k \in [0, 1)$. Let C be a bounded subset of a Hilbert space H. Let T be a k-strictly pseudo-contractive self-mapping on C. Let S be the mapping defined by $Sx = (kI + (1-k)T)x$ for $x \in C$. Assume that S is a self-mapping on C. Let $\{v_n\}$ and $\{b_n\}$ be sequences defined by $v_1 \in C$ and*

$$v_{n+1} = S v_n, \quad b_n = \frac{1}{n} \sum_{t=1}^{n} v_t \qquad \text{for} \quad n \in \mathbb{N}.$$

Then the followings hold.

(1) $\mathcal{A}_k(T)$ is non-empty, closed and convex.
(2) $\{b_n\}$ converges weakly to some $u \in \mathcal{A}_k(T)$.

Furthermore, if C is closed and convex then the followings hold.

(3) $F(T)$ is non-empty, closed and convex.
(4) $\{b_n\}$ converges weakly to $u \in F(T)$.

5. Strong convergence theorems for λ-hybrid mappings

In this section, we prove an attractive points theorem and strong convergence to attractive points of uniformly asymptotically regular λ-hybrid mappings in Hilbert spaces (see also [2, 7, 12, 18, 25, 26, 27, 29]).

Let C be a nonempty subset of H. Then, C is called star-shaped if there exists $z \in C$ such that for any $x \in C$ and any $\gamma \in (0, 1)$,

$$\gamma z + (1 - \gamma) x \in C.$$

We say that a mapping T of C into itself is asymptotically regular if

$$\lim_{n \to \infty} \|T^{n+1} x - T^n x\| = 0$$

for all $x \in C$ (see also [27]). We also say that a mapping T of C into itself is uniformly asymptotically regular if for every bounded subset K of C,

$$\lim_{n \to \infty} \sup_{x \in K} \|T^{n+1} x - T^n x\| = 0$$

holds.

Lemma 5.1 ([6]). *Let C be a nonempty subset of a Hilbert space H. Let $\lambda \in \mathbb{R}$ be given. Let T be a λ-hybrid mapping of C into itself. If $A(T) \neq \emptyset$, $\{T^n x\}$ is bouded for each $x \in C$.*

We also get the following attractive point theorems (see also [12, 29]).

Theorem 5.2 ([6]). *Let H be a Hilbert space and let C be a nonempty subset of H. Let λ be a real number. Let T be a uniformly asymptotically regular λ-hybrid mapping of C into itself. Suppose that $\{T^n x\}$ is bounded for some $x \in C$. Then, $A(T) \neq \emptyset$.*

We obtain a strong convergence theorem of Halpern's [21] type for λ-hybrd mappings on a star-shaped subset of H (see [6]).

Theorem 5.3 ([6]). *Let H be a Hilbert space, let C be a star-shaped subset of H with center $z \in C$. Let λ be a real number. Let T be a uniformly asymptotically regular λ-hybrid mapping of C into itself such that $A(T) \neq \emptyset$. Let $\{m_n\}$ be a sequence in $\mathbb{N}$ such that $m_n \to \infty$. Let $\{x_n\}$ be a sequence in C defined by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n z + (1-\alpha_n) T^{m_n} x_n$$

for each $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0,1]$ satisfies

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then, $\{x_n\}$ converges strongly to $P_{A(T)} z$, where $P_{A(T)}$ is the metric projection from H onto $A(T)$.

Using Theorem 5.2, we obtain the following fixed point theorem.

Theorem 5.4 ([6]). *Let H be a Hilbert space and let C be a closed and star-shaped subset of H. Let λ be a real number. Let T be a uniformly asymptotically regular λ-hybrid mapping of C into itself. Suppose that $\{T^n x\}$ is bounded for some $x \in C$. Then, $F(T) \neq \emptyset$.*

Using Theorem 5.3, we also get the following strong convergence theorem for λ-hybrid mappings on a star-shaped subset of H (see [21, 30, 31]).

Theorem 5.5 ([6]). *Let H be a Hilbert space, let C be a closed and star-shaped subset of H with center $z \in C$. Let λ be a real number. Let T*

be a uniformly asymptotically regular λ-hybrid mapping of C into itself such that $F(T) \neq \emptyset$. Let $\{m_n\}$ be a sequence in $\mathbb{N}$ such that $m_n \to \infty$. Let $\{x_n\}$ be a sequence in C defined by $x_1 \in C$ and

$$x_{n+1} = \alpha_n z + (1-\alpha_n)T^{m_n}x_n$$

for each $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0,1]$ satisfies

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then, $\{x_n\}$ converges strongly to u_0, where $\|u_0 - z\| = \min\{\|u - z\| : u \in F(T)\}$

We also have the following strong convergence theorem.

Theorem 5.6 ([6]). *Let H be a Hilbert space, let C be a nonempty subset of H. Let λ be a real number. Let T be a uniformly asymptotically regular λ-hybrid mapping of C into itself such that $A(T) \neq \emptyset$. Let $\{m_n\}$ be a sequence in $\mathbb{N}$ such that $m_n \to \infty$. Let $\{x_n\}$ be a sequence in C defined by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n z + (1-\alpha_n)T^{m_n}x_n$$

for each $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0,1]$ satisfies

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty.$$

If $\{x_n\}$ is in C, then $\{x_n\}$ converges strongly to $u_0 \in A(T)$, where $u_0 = P_{A(T)}$.

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